Why a square circle is an impossibility

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Some terminology

There is a kind of philosophical skepticism that rejects any claim of the form "X is not possible." This position has the appeal of apparent humility. It counsels us to be ever mindful of the limits of human knowledge and the folly of assuming an inverse relationship between certainty and fallibility.

But if, in striving to be humble, we reject logic, we attain no virtue thereby, either intellectual or moral.

Logic asserts that any meaningful statement or proposition P is either true or false, that it cannot be both and cannot be neither. Whether we can know which is the case is another matter, but if we think logically, we know that it must be one or the other.

And what is a meaningful proposition? It is one so stated that if its terms are well defined, then its truth or falsity is relevant to the context in which it appears. But that is just a noncircular way of saying that a meaningful statement is one that is either true or false and cannot be both.

Those uncomfortable with circular definitions are welcome to suggest alternatives. If a statement such as "The moon is blue" could be both true and false, or could be neither true nor false, then what possible meaning could it have? What could "The statement means X" itself even mean?

Likewise, the terms of a proposition are well defined if they allow a determination of the proposition's truth value. To do this, the definitions must be among other things unamibiguous, and every definition must remain constant within the course of any given argument.

By *argument*, we mean here any set of meaningful statements intended to demonstrate that one of them, called the conclusion, must be true if all the other statements, called premises, are simultaneously true. If the statements do so demonstrate, then we call the argument *valid*.

Suppose we have a set of premises P_1, P_2, ... P_n. Collectively we designate the set P, where P = (P_1, P_2, ... P_n). We say they prove the truth of a conclusion, C. The argument asserts that P *implies* C. By definition, we mean that it is not logically possible for C to be false if P is true.

It's time to bring in some symbolism. We'll try to keep it to a minimum.

Negation

~ means *negation*. Thus, if P is a statement, ~P is its negation.

Example: S = "Socrates is a man"; ~S = "Socrates is not a man." M = "All men are mortal"; ~M = "Not all men are mortal" = "Some men are not mortal" = "At least one man is not mortal."

Conjunction

$\wedge$ means AND, representing a *logical conjunction*. Thus, the conjunction of A and B is $A \wedge B$. A conjunction is true if and only if all its constituent statements are true. For any $P = (P_1, P_2, \ldots P_n)$, P is false is any of $(P_1, P_2, \ldots P_n)$ is false.

Example: P_1 = "Socrates is a man"; P_2 = "All men are mortal." Then let $P = P_1 \wedge P_2$. P is true if and only if Socrates is a man and all men are mortal. P is false if Socrates is not a man or if at least one man is immortal.

Disjunction

$\vee$ represents the *logical disjunction,* otherwise called OR. The disjunction of A and B is $A \vee B$. A disjunction is true if any of its constituent statements is true, and it is false only if all are false.

Example: P_1 = "Socrates is a man"; P_2 = "All men are mortal." Then let $C = P_1 \vee P_2$. C is true if Socrates is a man, even if some men are mortal. P is true if all men are mortal, even if Socrates is not a man. P is false if and only if "Socrates is not a man" and "at least one man is immortal" are both true.

Equivalence

<=> means *is logically equivalent to*. The relationship A <=> B exists if, whenever A is true, B is true and whenever A is false, B is false, and the converses hold.

Example: M = "All men are mortal"; N = "No man is immortal." Then M <=> N.

Implication

=> means *implies*. If A implies B, then it is not logically possible for B to be false while A is true.

Example:

S = "Socrates is a man"

M = "All men are mortal"

$P = S \wedge M$

C = "Socrates is mortal"

Then P => C, meaning that if C is false, then either Socrates is not a man, or else at least one man is not mortal.

Some additional observations

A logical disjunction is never exclusive unless explicitly stated. That is, $A \vee B$ is true if any of the following hold:

Only A is true

Only B is true

A and B are both true.

Thus:

$A \wedge \sim B \Rightarrow A \vee B$

$\sim A \wedge B \Rightarrow A \vee B$

$A \wedge B \Rightarrow A \vee B$.

but

$\sim A \wedge \sim B \Rightarrow \sim(A \vee B)$

Notice that any P => C does not mean P must be true for C to be true. It says only that C cannot be false if P is true.

Computer programming languages recognize a logical operator called the "exclusive OR," usually represented as XOR. A statement such as (A XOR B) is true if A is true, and true if B is true, but false if A and B are both true. No equivalent symbol has been adopted by philosophers. An exclusive disjunction must be spelled out like this: $[(A \lor B) \land \sim(A \land B)]$.

The fundamental notion of logic is that for any statement P:

$P \lor \sim P$ is always true

$P \land \sim P$ is never true

Or, given any P,

$$P \Rightarrow (P \lor \sim P) \land \sim(P \land \sim P)$$

The labeling of this notion varies. That $P \lor \sim P$ must be true is called the Law (or Principle, or Axiom) of the Excluded Middle. That $P \land \sim P$ is never true is sometimes called the Law (or etc.) of Noncontradiction and sometimes the Law of Contradiction. Some people are referring to both when they speak of the Law of the Excluded Middle. We will refer to $P \Rightarrow (P \lor \sim P) \land \sim(P \land \sim P)$ as the Fundamental Axiom of Logic, and not distinguish Noncontradiction from Excluded Middle unless it seems necessary to make a point.

(There is another fundamental notion, called the Law of Identity, which asserts that P = P. That one doesn't get much attention, for reasons that are probably obvious, but we would not be sufficiently thorough if we failed to mention it.)

Everything else that one learns in a formal study of logic is an elaboration of the Fundamental Axiom. A valid argument works by demonstrating that if the conclusion were false, then for at least one premise P_n of the argument, $(P_n \land \sim P_n)$ would have to be true.

To call an argument invalid is to say only that there would not necessarily be a contradiction, that the conclusion could be false even though all the premises were true. It is not to say that the conclusion has been proven false. It is only to say that it has not been proven true. By the Fundamental Axiom, we know it must be true or false, but an invalid argument cannot show us which it is.

Here is an invalid argument.

S = Socrates is a man.

M = Some men are mortal.

Therefore,

C = Socrates is mortal.

("Therefore" is shorthand for "The preceding statements logically imply the truth of the statement immediately following.")

The negation of C (~C = Socrates is not mortal) contradicts neither S nor M, because "some" is understood to mean "not necessarily all." The statement "Some men are mortal" is logically consistent with the statement "Some men are not mortal."

Of course the conclusion is true, but this argument does not prove it. If we wish to prove it, we must find another argument.

What about those premises?

In this section, "statement" is to be taken to mean possibly any conjunction of statements unless otherwise qualified as "simple statement."

Recall that a valid argument establishes that P => C where P is any set of statements, each of which is called a premise — $P = (P_1 \wedge P_2 \wedge \ldots \wedge P_n)$. To call it a valid argument is to say that IF all the premises are true, THEN the conclusion C must be true. That is to say, we are asserting that $(P \wedge \sim C)$ is logically impossible.

Of course the logical impossibility of $(P \wedge \sim C)$ does not matter much if P happens to be false, which it would be if any premise were false. If any statement in a conjunction is false, then the conjunction itself is false.

Logic has nothing to say about whether a particular premise is true. Logic is solely about the relationship between statements. It tells us whether, given any two statements A and B, the truth of one tells us anything about the truth of the other. Given the assertion A => B, we can examine the argument to see whether the conjunction $(A \wedge \sim B)$ is possible. If it is possible, then A => B is invalid. Otherwise, it is valid.

What if we happen to know that B is false? Then, if A => B is a valid argument, we have proven that at least one premise comprising A is false. If on the other hand we know that A cannot be false, then we know that neither can B be false. Given a valid argument, then, if we wish to determine whether the conclusion is necessarily true, we must determine whether the premises are true.

A valid argument with true premises is called a *sound* argument. A valid argument may be unsound, and if it is, then the conclusion is unproven. The conclusion could be true, but an argument using a false premise cannot prove it true. This must be emphasized. If P => C is valid, then we have shown that (P ∧ ~C) would be a contradiction. By the Fundamental Axiom, (P ∧ ~C) cannot be true if it is a contradiction, but if P happens to be false, then there is no contradiction.

If P => C is invalid, then the truth of P is irrelevant. If it is valid, then P must be examined for its truth or falsity. If it is false, then the validity of P => C is irrelevant.

Let's see how this works with Socrates. We have demonstrated that IF Socrates is a man AND IF all men are mortal, THEN Socrates is mortal. But have we demonstrated the truth of either premise?

No, not yet. At least, not in this essay, and not in any book I have read in which that syllogism has been discussed.

Are they not obviously true? Well, so is the conclusion, but then what is the point of the argument?

The point of the argument — any argument — is to show that we must believe the conclusion if we believe the premises (because if we don't, then we're believing a contradiction), but the argument cannot tell us whether we should believe the premises.

So, should we believe that all men are mortal? Well, the common experience of humanity is that all men die. Some adherents of certain religions will dispute that, but let us disregard their objections for the time being. Most of us do believe that all men die. Whether we are justified in that belief is an issue addressed by the branch of philosophy called epistemology.

What about Socrates being a man? How do we know that Socrates is not an angel?

Because nobody said he was. Unless context demands otherwise, words are to be understood in their usual sense if their usual sense is unambiguous. If context does demand otherwise, then a careful logician states his definitions before, or while, presenting his argument. If words are ambiguous in normal usage, then they should be avoided or their meanings must be clarified.

Logic is not about the search for absolute certainty. It only informs us that if some particular things are true, then certain other things must be true and certain other things cannot be true. It tells us that if we do not doubt that Socrates is a man and do not doubt that all men are mortal, then we are without justification for doubting that Socrates is mortal.

But for any set of statements used as premises for an argument, we can always inquire about their justification, and justification may hinge on our definitions of key terms.

Some premises can be justified on grounds of definition. "Socrates is a man" may be so justified. The word *Socrates* means a particular man who lived in Athens at a particular time. In that case, the premise “Socrates is a man” is *true by definition*.

Other premises might be justified as conclusions from other premises. Perhaps I have some argument of the form

Premise A

Premise B

Premise C

Therefore, all men are mortal.

I might then have other arguments proving those premises. At some point, I'm going to get to a set of premises that I cannot prove and that nobody else can prove, either, but that everyone agrees must be true anyway because nothing would make sense if they weren't true. Depending on context, such premises are called assumptions, postulates, or axioms.

Circles and squares

Euclidean geometry is derived from five premises that Euclid called postulates, plus a few others not specific to geometry, which he called axioms. It deals with entities such as points and lines, the existence of which is assumed and which, by definition, have certain properties.

The entities with which geometry deals do not exist in objective reality. They are mental constructs. There are real things, though, that are much like them in certain respects, and the theorems we deduce about the mental constructs are observed to be true of their real-world counterparts.

For instance, we can prove from Euclid's postulates that the two diagonals of any rectangle must be of equal length, where a rectangle is by definition a quadrilateral whose four vertices are all right angles. We cannot draw or manufacture anything with four sides and know that its angles are exactly 90 degrees, and neither can we know by measuring its diagonals that they are of exactly equal length. But we do observe that the theorem is confirmed within the limits of our ability to measure angles and line lengths.

This does not mean, though, that we have verified the theorem within the limits of human observation or experience and that some discovery tomorrow might prove Euclid was wrong about the diagonals of rectangles. Geometry is mathematics, and mathematics is neither proved nor disproved by observation. Every branch of mathematics begins with certain axioms for the sake of seeing what can be deduced from them. All of mathematics is basically just an exercise in logic. When the axioms happen to correspond well to some aspect of objective reality, then the deductions invariably turn out to correspond to reality just as well, and philosophers have entertained themselves for a long time trying to figure out why this is so.

The point here is that *within the Euclidean axiomatic system*, it is impossible, in an absolute sense, for a rectangle to have unequal diagonals. This is a simple logical implication. If a quadrilateral has unequal diagonals, then it cannot be a rectangle. Otherwise you'd have a contradiction. A rectangle's vertices are all equal, but unequal diagonals imply unequal vertices. It's that simple.

What if, in some situation not currently imaginable, the contradiction were observed: a quadrilateral with equal vertices but unequal diagonals? My guess would be that we should call an equiangular quadrilateral, in such a situation, something other than a rectangle. When mathematical logic leads us to expect certain things of reality, and reality then contradicts our expectations, then we need to revise not our mathematics but our expectations of reality.

Mathematically, a square circle is absolutely not possible because the definition of a circle and the definition of a square imply numerous contradictory properties. Some examples:

For any circle, there exists one point from which all points on the circle are equidistant. For any square, there exists no point from which all points on the square are equidistant.

- A square consists of exactly four line segments. A circle consists of an infinite number of line segments.

- A circle has a constant nonzero curvature at all points. A square has zero curvature at all points except the vertices, where the curvature is undefined.

- Through any point on a circle, exactly one tangent line may be drawn. Through any vertex of a square, an infinite number of tangent lines may be drawn, but through any other point no tangent line may be drawn.

But that is just the mathematical construct. Could reality someday surprise us with a square circle? Does our inability to imagine one prove anything about reality, or just about the limits of our imagination?

Reality is surely not bound by our imaginations, but it does seem to be bound by logic. A square circle in some sense goes beyond a rectangle with unequal diagonals. We could at least recognize something as a rectangle if we saw a four-sided figure with four equal vertices, and then be surprised if we measured its diagonals and found them unequal. But what in any conceivable universe could we look at that would exhibit any property characteristic of circles plus any other property characteristic of squares?

One may argue that that which is inconceivable is not therefore impossible, but any universe with a square circle would be a universe without any rules at all. In such a universe, there would be no difference between existence and nonexistence. There would be no difference between zero and any number, or between finite and infinite. There would be neither truth nor falseness.

We do not live in such a universe, and so there are no square circles and there can be none, either in theory or in fact.

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